JOURNAL OF APPROXIMATION THEORY 23, 354–363 (1978)

Degree of L_p-Approximation by Certain Positive Convolution Operators

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Communicated by Oved Shisha

Received December 21, 1976

The degree of L_p-approximation for a class of positive convolution operators is investigated. Recent results of De Vore, Bojanic, and Shisha for the uniform approximation by these operators and the K-functional of Peetre are employed to obtain the degree of approximation in terms of the integral modulus of smoothness.

1. INTRODUCTION

Let $I = [0, r]$, $1 \leqslant p < \infty$ and let $L_p(I)$ denote the space of real valued pth power integrable functions on I, with $\|\cdot\|_p$ the usual L_p-norm on I. Let $\{H_n(y)\}$ be a sequence of nonnegative, even and continuous functions on $[-r, r]$ such that

$$\int_{-r}^{r} H_n(y)\,dy = 1, \qquad n = 1, 2,..., \tag{1.1}$$

and

$$\int_{-r}^{r} y^2 H_n(y)\,dy \equiv \mu_n{}^2 \to 0, \qquad n \to \infty. \tag{1.2}$$

For $f \in L_p(I)$, $1 \leqslant p < \infty$, and $0 \leqslant x \leqslant r$, we define the convolution operator

$$K_n(f, x) = \int_0^r f(t)\, H_n(t - x)\,dt, \qquad n = 1, 2,... . \tag{1.3}$$

This is a linear operator mapping $L_p(I)$ into $L_p(I)$ and it is positive on I. The sequence $\{\mu_n\}$ determines the rate of uniform approximation of a continuous function f by the operator $K_n(f, x)$. There are two important examples of (1.3).

0021-9045/78/0234–0354\$02.00/0

Korovkin operators. Let ϕ be a nonnegative, even and continuous function on $[-r, r]$ decreasing on $[0, r]$ and such that $\phi(0) = 1$ and $0 \leqslant \phi(t) < 1$ for $0 < t \leqslant r$. For $f \in L_p(I)$ we define

$$K_n(f, x) = \rho_n \int_0^r f(t)[\phi(t - x)]^n \, dt, \tag{1.4}$$

where

$$\rho_n^{-1} = 2 \int_0^r [\phi(t)]^n \, dt.$$

Operators (1.4) were introduced by Korovkin [8], who used them for the approximation of continuous functions. Later Bojanic and Shisha [3] showed that

$$\lim_{t \to 0^+} \frac{1 - \phi(t)}{t^\alpha} = c \tag{1.5}$$

for some positive numbers α and c implies, for (1.4)

$$\mu_n = O(n^{-1/\alpha}), \qquad n \to \infty. \tag{1.6}$$

Many important special cases of (1.5) were noted in [3].

Bojanic–DeVore operators. We consider approximating polynomials generated by a sequence of orthogonal polynomials $\{P_n\}$ on $[-1, 1]$ whose weight function w is nonnegative, even and Lebesgue integrable on $[-1, 1]$ and has the following properties:

$$0 < m \leqslant w(x) \quad \text{for} \quad x \in [-r, r], \quad 0 < r \leqslant 1 \tag{1.7}$$

and

$$w(x) \leqslant M < \infty \quad \text{for} \quad x \in [-\delta, \delta], \quad 0 < \delta \leqslant 1. \tag{1.8}$$

Let

$$R_n(x) = c_n \left[\frac{P_{2n}(x)}{(x^2 - \alpha_{2n}^2)(x^2 - \alpha_{2n-1}^2)} \right]^2,$$

where α_{2n} and α_{2n-1} are the two smallest positive zeros of P_{2n} and $c_n > 0$ is chosen so that

$$\int_{-r}^{r} R_n(t) \, dt = 1, \qquad n = 1, 2, 3, \ldots .$$

For $f \in L_p(I)$ we define

$$K_n(f, x) = \int_0^r f(t) \, R_n(t - x) \, dt. \tag{1.9}$$

Using (1.7) and (1.8), Bojanic [2] and DeVore [4] showed that, for (1.9)

$$\mu_n = O(n^{-1}), \qquad n \to \infty. \tag{1.10}$$

In this paper we shall utilize Peetre's K-functional [12] to obtain the degree of L_p-approximation with (1.3). The method of Peetre's K-functional gives the estimate of the rate of approximation in terms of the integral modulus of smoothness $\omega_{2,p}(f, h)$, while some previous results in this direction were expressed in terms of the usual L_p modulus of continuity (see for instance, Mamedov [9]).

2. Degree of Approximation

In the sequel let $e_i(x) = x^i$ for $i = 0, 1, 2$.

Lemma 1. For $n = 1, 2, 3,\ldots$, we have

$$\int_0^r H_n(t - x)\, dt = K_n(e_0, x) \leqslant 1, \qquad 0 \leqslant x \leqslant r, \tag{2.1}$$

$$\int_0^r H_n(t - x)\, dx \leqslant 1, \qquad 0 \leqslant t \leqslant r, \tag{2.2}$$

$$\| K_n \|_p \leqslant 1, \qquad 1 \leqslant p < \infty, \tag{2.3}$$

$$| K_n((t - x), x)| \leqslant \frac{\mu_n^2}{\delta}, \qquad 0 < \delta \leqslant x \leqslant r - \delta < r, \tag{2.4}$$

$$| K_n(e_0, x) - 1 | \leqslant \frac{2\mu_n^2}{\delta^2}, \qquad 0 < \delta \leqslant x \leqslant r - \delta < r, \tag{2.5}$$

$$K_n((t - x)^2\, x) \leqslant \mu_n^2, \qquad 0 \leqslant x \leqslant r, \tag{2.6}$$

and

$$K_n(| t - x |, x) \leqslant \mu_n, \qquad 0 \leqslant x \leqslant r. \tag{2.7}$$

Proof. (2.1) For $x \in [0, r]$,

$$K_n(e_0, x) = \int_0^r H_n(t - x)\, dt \leqslant \int_{-r}^r H_n(y)\, dy = 1.$$

(2.2) For $t \in [0, r]$,

$$\int_0^r H_n(t - x)\, dx = \int_{t-r}^t H_n(y)\, dy \leqslant 1.$$

(2.3) Assume $p > 1$, $1/p + 1/q = 1$, $0 \leqslant x \leqslant r$, and $f \in L_p[0, r]$. By (2.1), (2.2), and Hölder's

$$|K_n(f, x)| \leqslant \left(\int_0^r H_n(t - x)\, dt\right)^{1/q} \left(\int_0^r H_n(t - x)\, |f(t)|^p\, dt\right)^{1/p}$$
$$\leqslant \left(\int_0^r H_n(t - x)\, |f(t)|^p\, dt\right)^{1/p}$$

and

$$\| K_n(f)\|_p \leqslant \|f\|_p .$$

The case $p = 1$ is similar.

(2.4) Since $H_n(y)$ is even on $[-r, r]$, for $0 < \delta \leqslant x \leqslant r - \delta < r$ we have

$$|K_n((t - x), x)| = \left| \int_0^r (t - x)\, H_n(t - x)\, dt \right|$$
$$= \left| \int_x^{r-x} y H_n(y)\, dy \right| \leqslant \frac{\mu_n^2}{\delta} .$$

(2.5) If $0 \leqslant x \leqslant r$ then

$$K_n(e_0, x) = 1 - \left[\int_{r-x}^r H_n(t)\, dt + \int_{-r}^{-x} H_n(t)\, dt\right].$$

If $0 < \delta \leqslant x \leqslant r - \delta < r$ then

$$| K_n(e_0, x) - 1 | \leqslant \frac{2}{\delta^2} \int_{-r}^r t^2 H_n(t)\, dt = 2\, \frac{\mu_n^2}{\delta^2} .$$

(2.6) This is obvious.

(2.7) This follows from Hölder's, (2.1) and (2.6).

Let $1 \leqslant p < \infty$ and $L_p^2(I)$ be the space of those functions $f \in L_p(I)$ with f' absolutely continuous and $f'' \in L_p(I)$. The next lemma gives an upper bound for the degree of L_p-approximation with (1.3) to "smooth" functions $f \in L_p^2(I)$.

LEMMA 2. *Let $f \in L_p^2(I)$ and $1 \leqslant p < \infty$. For all n sufficiently large,*

$$\| K_n(f) - f\|_p \leqslant c_p(\|f\|_p + \|f''\|_p)\, \mu_n^{1/p},$$

where c_p is a positive constant, independent of f and n.

Proof. First assume $p > 1$ and $x \in [0, r]$. Since $f \in L_p^2(I)$, we have

$$f(t) - f(x) = (t - x) f'(x) + \int_x^t (t - u) f''(u)\, du$$

and

$$K_n(f(t) - f(x), x) = f'(x)\, K_n(t - x, x) + K_n\left[\int_x^t (t - u) f''(u)\, du, x\right]$$
$$\equiv I_1(x) + I_2(x).$$

Using [6] we obtain

$$\| I_1 \|_p \leqslant c_1(\|f\|_p + \|f''\|_p)\left(\int_0^r | K_n(t - x, x)|^p\, dx\right)^{1/p}.$$

Also

$$| K_n(t - x, x)| = \left| \int_0^r (t - x)\, H_n(t - x)\, dt \right|$$
$$= \left| \int_{-x}^{r-x} tH_n(t)\, dt \right| \leqslant r, \qquad 0 \leqslant x \leqslant r.$$

Hence, using (2.4) with $0 < \delta < r/2$

$$\left(\int_0^r | K_n(t - x, x)|^p\, dx\right)^{1/p} \leqslant 2r\delta^{1/p} + r^{1/p} \frac{\mu_n{}^2}{\delta}$$

and

$$\| I_1 \|_p \leqslant c_2(\|f\|_p + \|f''\|_p)\left(\delta^{1/p} + \frac{\mu_n{}^2}{\delta}\right).$$

We have

$$| I_2(x)| \leqslant \int_0^r H_n(t - x)\, | t - x |\, \left| \int_x^t |f''(u)|\, du \right| dt$$
$$\leqslant \theta_{f''}(x) \int_0^r H_n(t - x)(t - x)^2\, dt,$$

where

$$\theta_{f''}(x) = \sup_{\substack{0 \leqslant t \leqslant r \\ t \neq x}} \frac{1}{t - x} \int_x^t |f''(u)|\, du$$

is the Hardy–Littlewood majorante of f'' at x. Since $p > 1$ and $f'' \in L_p(I)$, $\theta_{f''} \in L_p(I)$ with [13, Theorem 13.15]

$$\int_0^r [\theta_{f''}(x)]^p\, dx \leqslant 2\left(\frac{p}{p - 1}\right)^p \int_0^r |f''(x)|^p\, dx.$$

Therefore, using (2.6)

$$\| I_2 \|_p \leqslant 2^{1/p}\left(\frac{p}{p - 1}\right) \|f''\|_p\, \mu_n{}^2.$$

By [6]

$$|f(x)|\, | K_n(e_0, x) - 1 | \leqslant C_3(\|f\|_p + \|f''\|_p)\, | K_n(e_0, x) - 1 |.$$

Let $0 < \delta < r/2$. Using (1.1), (2.1), (2.5) and the fact that H_n is even we have

$$\begin{aligned}
&\| K_n(e_0) - e_0 \|_p \\
&\quad \leqslant \left(\int_0^r (1 - K_n(e_0, x)\, dx \right)^{1/p} \\
&\quad = \left[\left(\int_0^\delta + \int_\delta^r \right) \int_x^r H_n(t)\, dt\, dx + \left(\int_0^{r-\delta} \int_{r-\delta}^r \right) \int_{r-x}^r H_n(t)\, dt\, dx \right]^{1/p} \\
&\quad \leqslant \left[\delta + \int_\delta^r H_n(t) \int_0^t dx\, dt + \delta + \int_\delta^r H_n(t) \int_{r-t}^r dx\, dt \right]^{1/p} \\
&\quad \leqslant \left[2 \left(\delta + \frac{\mu_n^2}{\delta} \right) \right]^{1/p}.
\end{aligned}$$

Hence

$$\left(\int_0^r (|f(x)|\, |\, K_n(e_0, x) - 1\, |)^p\, dx \right)^{1/p} \leqslant c_4(\|f\|_p + \|f''\|_p) \left(\delta + \frac{\mu_n^2}{\delta} \right)^{1/p}.$$

Choose $\delta = \mu_n$. For all n sufficiently large, $0 < \mu_n < r/2$ by (1.2) and it follows from the above calculations that

$$\| K_n(f) - f \|_p \leqslant C_p(\|f\|_p + \|f''\|_p)\, \mu_n^{1/p}.$$

Now assume $p = 1$, $x \in [0, r]$ and $0 < \delta < r/2$. As before,

$$\int_0^r |f(x)|\, |\, K_n(e_0, x) - 1\, |\, dx \leqslant C_5(\|f\|_1 + \|f''\|_1) \left(\delta + \frac{\mu_n^2}{\delta} \right)$$

and

$$\int_0^r |f'(x)|\, |\, K_n(t - x, x)|\, dx \leqslant C_6(\|f\|_1 + \|f''\|_1) \left(\delta + \frac{\mu_n^2}{\delta} \right).$$

Next

$$\begin{aligned}
&\int_0^r |\, I_2(x)|\, dx \\
&\quad \leqslant \int_0^r \int_0^r H_n(t - x)\, |\, t - x\, |\, \left| \int_x^t |f''(u)|\, du \right| dt\, dx \\
&\quad = \left(\int_0^\delta + \int_\delta^{r-\delta} + \int_{r-\delta}^r \right) \int_0^r H_n(t - x)\, |\, t - x\, |\, \left| \int_x^t |f''(u)|\, du \right| dt\, dx \\
&\quad \equiv A_1 + A_2 + A_3.
\end{aligned}$$

For A_2 let $T_1 = \{t: |t - x| \leqslant \delta\}$ and $T_2 = \{t: |t - x| \geqslant \delta\}$. Then, using (2.6),

$$A_2 \leqslant \int_\delta^{r-\delta} \int_{T_1} H_n(t - x) \, |x - t| \left| \int_x^t |f''(u)| \, du \right| dt \, dx$$

$$+ \int_\delta^{r-\delta} \int_0^r H_n(t - x) \frac{(t - x)^2}{\delta} \left| \int_x^t |f''(u)| \, du \right| dt \, dx$$

$$\leqslant \delta \int_\delta^{r-\delta} \int_{T_1} H_n(t - x) \left| \int_x^t |f''(u)| \, du \right| dt \, dx + r \, \|f''\|_1 \frac{\mu_n^2}{\delta}.$$

Also

$$\int_\delta^{r-\delta} \int_{T_1} H_n(t - x) \left| \int_x^t |f''(u)| \, du \right| dt \, dx$$

$$\leqslant \int_\delta^{r-\delta} \left\{ \int_{x-\delta}^x H_n(t - x) \int_t^x |f''(u)| \, du \, dt \right.$$

$$\left. + \int_x^{x+\delta} H_n(t - x) \int_x^t |f''(u)| \, du \, dt \right\} dx$$

$$\leqslant \int_\delta^{r-\delta} \left\{ \int_{x-\delta}^x H_n(t - x) \int_{x-\delta}^x |f''(u)| \, du \, dt \right.$$

$$\left. + \int_x^{x+\delta} H_n(t - x) \int_x^{x+\delta} |f''(u)| \, du \, dt \right\} dx$$

$$\leqslant \int_\delta^{r-\delta} \left\{ \int_{x-\delta}^x |f''(u)| \int_0^r H_n(t - x) \, dt \, du \right.$$

$$\left. + \int_x^{x+\delta} |f''(u)| \int_0^r H_n(t - x) \, dt \, du \right\} dx$$

$$\leqslant \int_\delta^{r-\delta} \int_{x-\delta}^{x+\delta} |f''(u)| \, du \, dx = \int_\delta^{r-\delta} \int_{-\delta}^{\delta} |f''(t + x)| \, dt \, dx$$

$$= \int_{-\delta}^{\delta} \int_\delta^{r-\delta} |f''(t + x)| \, dx \, dt \leqslant 2\delta \, \|f''\|_1 .$$

Hence

$$A_2 \leqslant \|f''\|_1 \left(r \frac{\mu_n^2}{\delta} + 2\delta^2 \right).$$

Finally, using (2.7)

$$A_i \leqslant \|f''\|_1 \, \delta \mu_n , \qquad i = 1, 3,$$

and therefore

$$\int_0^r |I_2(x)|\,dx \leqslant C_7 \|f''\|_1 \left(\delta^2 + \delta\mu_n + \frac{\mu_n^2}{\delta}\right).$$

Lemma 2 for $p = 1$ follows by choosing $\delta = \mu_n$.

In what follows we will measure smoothness using the K-functional of Petre [12]. It is, for $f \in L_p(I)$, $1 \leqslant p \leqslant \infty$, defined by

$$K_p(t, f) = \inf_{g \in L_p^2(I)} (\|f - g\|_p + t(\|g\|_p + \|g''\|_p)), \qquad t \geqslant 0.$$

This measures the degree of approximation of a function $f \in L_p(I)$ by smoother functions $g \in L_p^2(I)$ with simultaneous control on the size of $\|g\|_p + \|g''\|_p$. The second order integral modulus of smoothness is given by

$$\omega_{2p}(f, h) = \sup_{0<t\leqslant h} \|f(\cdot + t) - 2f(\cdot) + f(\cdot - t)\|_p\,(I_t)$$

where I_t indicates that the L_p-norm is taken over the interval $[0 + t, r - t]$. It is known [5, 7] that there are constants $c_1 > 0$, $c_2 > 0$, independent of f and p, such that

$$c_1\omega_{2,p}(f, t^{1/2}) \leqslant K_p(t, f) \leqslant \min(1, t)\|f\|_p + 2c_2\omega_{2,p}(f, t^{1/2}). \tag{2.8}$$

We shall use Lemma 2 and (2.8) to establish the degree of approximation with (1.3). Namely, we first approximate $f \in L_p(I)$ by $g \in L_p^2(I)$ and then use Lemma 2, the definition of the K-functional and (2.8). See also [1, 5, 10, 11] for this approach.

THEOREM 3. *Let $f \in L_p(I)$, $1 \leqslant p < \infty$. For all n sufficiently large,*

$$\|K_n(f) - f\|_p \leqslant M_p[\mu_n^{1/p}\|f\|_p + \omega_{2,p}(f, \mu_n^{1/2p})], \tag{2.9}$$

where M_p is a positive constant, independent of f and n.

Proof. For all n sufficiently large,

$$\begin{aligned} \|K_n(h) - h\|_p &\leqslant 2\|h\|_p, & h \in L_p(I), \\ &\leqslant C_p\mu_n^{1/p}(\|h\|_p + \|h''\|_p), & h \in L_p^2(I), \end{aligned}$$

where C_p is positive constant, independent of h and n. When $f \in L_p(I)$ and g is an arbitrary function in $L_p^2(I)$, then

$$\begin{aligned} \|K_n(f) - f\|_p &\leqslant \|K_n(f - g) - (f - g)\|_p + \|K_n(g) - g\|_p \\ &\leqslant 2\|f - g\|_p + C_p\mu_n^{1/p}(\|g\|_p + \|g''\|_p). \end{aligned}$$

Taking the infimum over all $g \in L_p^2(I)$ on the right hand side, using the definition of the K-functional and (2.8), we obtain (2.9).

COROLLARY 4. *If* $f \in \mathrm{Lip}(\beta, L_p(I))$ *for* $0 < \beta \leqslant 1$ *then*

$$\| K_n(f) - f \|_p = 0(\mu_n^{\beta/2p}), \qquad n \to \infty.$$

Here the Lipschitz class $\mathrm{Lip}(\beta, L_p)$ of order β with respect to the L_p-norm is defined as the collection of all functions $f \in L_p(I)$ with the property $\omega_{2,p}(f, t) = O(t^\beta)$, $t \to 0^+$.

Remarks. Since (1.3) is a global contractive map, we can also apply the general quantitative estimate of Berens and De Vore [1]. Define

$$\lambda_{n,p}^2 = \max_{i=0,1} \| K_n(e_i) - e_i \|_p .$$

We have shown in the proof of Lemma 2 that $\| K_n(e_0) - e_0 \|_p = O(\mu_n^{1/p})$ as $n \to \infty$. Also

$$\| K_n(e_1) - e_1 \|_p \leqslant \int_0^r | K_n(t - x, x)|^p \, dx^{1/p} + r \| K_n(e_0) - e_0 \|_p .$$

It now follows from (2.4) that $\| K_n(e_1) - e_1 \|_p = O(\mu_n^{1/p})$ as $n \to \infty$. Hence

$$\lambda_{n,p}^2 = 0(\mu_n^{1/p}), \qquad n \to \infty.$$

Applying [1, p. 291] we obtain, for all sufficiently large n,

$$\| K_n(f) - f \|_p \leqslant M_p'[\mu_n^{1/p^2} \| f \|_p + \omega_{2,p}(f, \mu_n^{1/2p^2})], \tag{2.10}$$

where M_p' is a positive constant, independent of f and n.

Estimate (2.9) is better than (2.10) if $p > 1$. This is not too surprising, since Lemma 1 of [1] appears to be a rather coarse estimate for $p > 1$.

ACKNOWLEDGMENTS

The author is grateful to Professor John Swetits of Old Dominion University and the referee for several helpful comments.

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